SCORE: 8 / 30 POINTS

1. NO CALCULATORS ALLOWED
2. UNLESS STATED OTHERWISE, YOU MUST SIMPLIFY ALL ANSWERS
3. SHOW PROPER CALCULUS LEVEL WORK TO JUSTIFY YOUR ANSWERS

Determine if the sequence $a_n = n \sin \frac{1}{n}$ converges or diverges. If it converges, find its limit. Justify your answer using proper algebra and/or calculus. State your conclusion clearly.

SCORE: $\frac{1}{2}$ / 5 PTS

$$a_n = n \sin \frac{1}{n}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n \sin \frac{1}{n}$$

$$= \lim_{n\to\infty} n \cdot \lim_{n\to\infty} \sin \frac{1}{n}$$

$$= 0$$

$\therefore$ the sequence converges

$\frac{1}{2}$

Determine if $\left\{ \dfrac{n^2}{\sqrt{n^3+4}} \right\}$ converges or diverges. If it converges, find its limit.

SCORE: $3\frac{1}{2}$ / 4 PTS

Justify your answer using proper algebra and/or calculus. State your conclusion clearly.

$$a_n = \left\{ \frac{n^2}{\sqrt{n^3+4}} \right\}$$

$$\lim_{n\to\infty} \left\{ \frac{n^2}{\sqrt{n^3+4}} \right\}$$

$$= \lim_{n\to\infty} \frac{n^2}{\sqrt{n^3+4}} = \lim_{n\to\infty} \frac{n^2/\sqrt{n^3}}{\sqrt{n^3+4}\,/\sqrt{n^3}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{1+4/n^3}} = \frac{\lim_{n\to\infty} \sqrt{n}}{\lim_{n\to\infty} \sqrt{1+4/n^3}} = \frac{\infty}{1} = \infty$$

$\therefore$ the sequence converges

$1\frac{1}{2}$ 1   1

Consider the sequence defined recursively by $a_1 = 3, \quad a_{n+1} = \dfrac{a_n}{1+a_n}$. **SCORE:** 3 **/ 3 PTS**

[a] List the first 4 terms of the sequence. Write your final answer as a list.

$a_1 = 3$

$a_{n+1} = \dfrac{a_n}{1+a_n}$

$a_1 + 1 = \dfrac{a_1}{1+a_1}$

$a_2 = \dfrac{3}{1+3} = \dfrac{3}{4}$

$a_{2+1} = \dfrac{a_2}{1+a_2}$

$a_3 = \dfrac{3/4}{1+3/4}$

$a_3 = \dfrac{3/4}{7/4} = \dfrac{3}{7}$

$a_{3+1} = \dfrac{a_3}{1+a_3} = \dfrac{3/7}{1+3/7} = \dfrac{3/7}{10/7} = \dfrac{3}{10}$

$a_4 = 3/10$

$a_{4+1} = \dfrac{a_4}{1+a_4}$

$a_5 = \dfrac{3/10}{1+3/10} = \dfrac{3/10}{13/10} = \dfrac{3}{13}$

$\therefore a_1 = 3,\ a_2 = \dfrac{3}{4},\ a_3 = \dfrac{3}{7},\ a_4 = \dfrac{3}{10},\ a_5 = \dfrac{3}{13}$

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[b] Find a formula for the general term a_n of the sequence, assuming that the pattern of the first four terms from [a] continues.

$a_n = \{3, 3/4, 3/7, 3/10, 3/13, \cdots\}$ $\qquad 3, 3/4, 3/7, 3/10, 3/13$

$a_n = \dfrac{3}{3n-2}$

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Determine if the sequence $a_n = \dfrac{\cos^2 n}{2^n}$ converges or diverges. If it converges, find its limit. **SCORE:** 0 **/ 4 PTS**

Justify your answer using proper algebra and/or calculus. State your conclusion clearly.

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{\cos^2 n}{2^n} = \dfrac{\lim_{n\to\infty} \cos^2 n}{\lim_{n\to\infty} n \ln 2}$

when $\lim\limits_{n\to\infty} \cos^2 n = \lim\limits_{n\to\infty} (\cos n)^2 = (\cos 0)^2 = 1$

$\lim\limits_{n\to\infty} n \ln 2 = \infty$

$\therefore$ the sequence diverges

Determine if the sequence defined recursively by $a_1 = 2, \quad a_{n+1} = \frac{1}{2}(6 + a_n)$ is increasing or decreasing. **SCORE:** 1 **/ 4 PTS**

Use mathematical induction to prove your answer. State your conclusion clearly.

$a_{n+1} = \frac{1}{2}(6 + a_n)$

$a_1 = 2$

$a_2 = \frac{1}{2}(6 + a_1) = \frac{1}{2}(6+2) = 4$

$a_3 = \frac{1}{2}(6 + a_2) = \frac{1}{2}(6+4) = 5$

$a_4 = \frac{1}{2}(6 + a_3) = \frac{1}{2}(6+5) = \frac{11}{2} = 5.5$

$a_5 = \frac{1}{2}(6 + a_4) = \frac{1}{2}(6 + \frac{11}{2}) = \frac{1}{2}(\frac{23}{2}) = \frac{23}{4} \approx 5.75$

$a_6 = \frac{1}{2}(6 + a_5) = \frac{1}{2}(6 + \frac{23}{4}) = \frac{1}{2}(\frac{47}{4}) = \frac{47}{8}$

The sequence is increasing.

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